1.2 Noetherian/Artinian Modules and Rings

R ring.

Lemma 1.1 For $M \in \text{Mod-}R$, TFAE

(a) Every $N_R \leq M_R$ is finitely generated
submodule

(b) M satisfies the ascending chain condition (ACC) on submodules. I.e., if $M_1 \subseteq M_2 \subseteq \dots$ are submodules, there exists $n_0 \geq 1$ s.t. $\forall n \geq n_0$: $M_n = M_{n_0}$.

(c) Every nonempty set Ω consisting of submodules of M has a maximal element.

Proof: (a) $\Rightarrow$ (b) $M' := \bigcup_{i \geq 1} M_i$ is a submodule of M, hence

$\exists m_1, \dots, m_k \in M'$: $M' = \langle m_1, \dots, m_k \rangle_R$ $(= m_1 R + \dots + m_k R)$ by (a).

$\Rightarrow \exists n_0 \geq 0$: $m_1, \dots, m_k \in M_{n_0} \Rightarrow M' \subseteq M_{n_0} \subseteq M_n \subseteq M'$ $\forall n \geq n_0$

$\Rightarrow M_n = M_{n_0}$.

(b) $\Rightarrow$ (c) Suppose not. Then: $\forall N \in \Omega\ \exists N' \in \Omega$: $N \subsetneq N'$.

Choose $N_0 \in \Omega$ arbitrary. Recursively, $\forall i \geq 0$, choose $N_{i+1} \in \Omega$ s.t.

$N_i \subsetneq N_{i+1}$ $\lightning$.

(c) $\Rightarrow$ (a) Let $N \leq M$. Let $\Omega := \{ N' \leq N : N' \text{ is f.g.} \}$.

Then $0 \in \Omega \Rightarrow \Omega \neq \emptyset$. So $\exists N_0 \in \Omega$ that is maximal.

If $N_0 \subsetneq N$, then $\exists x \in N \setminus N_0$, and $N_0 + xR \supsetneq N_0$ but $N_0 + xR \in \Omega$ $\lightning$.

So $N = N_0$ is f.g. $\square$

Def (1) $M \in \text{Mod-}R$ is noetherion if it satisfies the conditions in Lemma 1.1

(2) R is right [left] noetherion if R_R [${}_RR$] is noetherion.

(3) R is noetherion if it is right and left noetherion.

So: R right noeth. $\iff$ every right ideal is f.g.

Exm: •) $\mathbb{Z}$ is noetherion; R noetherion $\Rightarrow R[x_1, \ldots, x_n]$ noetherion (Hilbert's basis theorem).

•) Free algebras in ≥ 2 variables are not noetherion.

E.g. $F = k\langle x, y\rangle \Rightarrow \sum_{i \geq 0}' x^i y F$ is direct (Exercise), hence not f.g.

•) $R = k(x)[y; \sigma]$ with $\sigma\left(\frac{f}{g}\right) = \frac{f(x^2)}{g(x^2)}$ is left noetherion, not right noetherion (Exercise: left PID by polynomial division; but $\sum_{i \geq 0}' y^i x y R$ is direct)

Prop 1.2: Let $\sigma \in \mathrm{Aut}(R)$ (ring automorphisms), δ a σ-derivation. If R is right [left] noetherion, then $R[x; \sigma, \delta]$ is right [left] noetherion.

Proof omitted, like commutative Hilbert's basis theorem (i.e, for left noetherion: given $L \leq_R R$, consider the left ideals $L_n \leq_R R$ consisting of leading coeffs of polynomial of degree $\leq n$ in L + zero)

Lemma 1.3 For $M \in \text{Mod-}R$, TFAE:

(a) M satisfies the descending chain condition (DCC) on submodules, i.e., for every chain $M_1 \supseteq M_2 \supseteq \ldots$ of submodules, there exists n_0 s.t. $\forall n \geq n_0$: $M_{n_0} = M_n$.

(b) Every non-empty set consisting of submodules of M has a minimal element.

Proof: (a) $\Rightarrow$ (b): Analogous to L1.1.(b) $\Rightarrow$ (c)

(b) $\Rightarrow$ (a) Let $M_1 \supseteq M_2 \supseteq \ldots$ be a descending chain, $\Omega := \{M_i : i \geq 1\}$

$\Rightarrow \Omega$ has a minimal element $M_{n_0} \Rightarrow \forall n \geq n_0$, $M_n = M_{n_0}$. $\square$

Def: (1) $M \in \text{Mod-}R$ is artinian if it satisfies the conditions of Lemma 1.3.

(2) R is right [left] artinian if R_R [${}_RR$] is artinian.

(3) R is artinian if it is right and left artinian.

Exm: • Division rings are artinian, $\mathbb{Z}/n\mathbb{Z}$, $n \neq 0$ is artinian, $\mathbb{Z}$ is not artinian ($\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supsetneq 8\mathbb{Z} \supsetneq \ldots$)

• $\begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$ is right noetherian not left noetherian

• $\begin{bmatrix} \mathbb{Q} & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$ is right artinian, not left artinian (Exercise, cf. [Lam01, §1])

Lemma 1.4 (1) Let $M \in \text{Mod-}R$, $N \leq M$. Then

M is noetherian [artinian] $\Leftrightarrow$ N and M/N are noetherian [artinian]

(2) Let $M_1, \ldots, M_n \in \text{Mod-}R$. Then $\bigoplus_{i=1}^{n} M_i$ is noetherian [artinian]

$\Leftrightarrow \forall i$: M_i is noetherian [artinian].

(1) restated: if $0 \to N \to M \to \overset{\cong M/N}{L} \to 0$ is a short exact sequence (SES), then M noetherian [artinian] $\Leftrightarrow$ N, L noetherian [artinian]

Proof: (1) For artinian, noetherian is dual.

„$\Rightarrow$": If $N_1 \supseteq N_2 \supseteq \dots$ is a chain of submodules of N, then this is also a chain of submodules of M, so stabilizes.

Suppose $L_1 \supseteq L_2 \supseteq \dots$ are submodules of M/N. Then $L_i = M_i+N/N$ with M_i submodules of M, $M_1+N \supseteq M_2+N \supseteq \dots$, so

$\exists i_0 \forall i \geq i_0: M_i = M_{i_0} \Rightarrow M_i + N = M_{i_0} + N \Rightarrow L_i = L_{i_0}$.

„$\Leftarrow$" Let $M_1 \supseteq M_2 \supseteq \dots$ be submodules of M.

Both chains $M_1 \cap N \supseteq M_2 \cap N \supseteq \dots$ and

$M_1+N/N \supseteq M_2+N/N \supseteq \dots$ stabilize

$\Rightarrow \exists i_0 \forall i \geq i_0$: $M_i \cap N = M_{i_0} \cap N$ and $M_i+N/N = M_{i_0}+N/N$. $\Rightarrow M_i+N = M_{i_0}+N$

Claim: $M_i = M_{i_0}$.

„$\subseteq$" ✓ „$\supseteq$" Let $m \in M_{i_0} \Rightarrow m \in M_{i_0}+N = M_i+N$, so $m = m'+n$ ($m' \in M_i$, $n \in N$)

$\xRightarrow{M_i \subseteq M_{i_0}} n = m - m' \in M_{i_0}$ and also $n \in N \xRightarrow{M_{i_0} \cap N = M_i \cap N} n \in M_i. \Rightarrow m = \underset{M_i}{\underset{\cap}{m'}} + \underset{M_i}{\underset{\cap}{n}} \in M_i$

(2) By (1) and induction, because there is a S.E.S.

$$0 \to \bigoplus_{i=1}^{n-1} M_i \to \bigoplus_{i=1}^{n} M_i \to M_n \to 0.$$

□

2. Semisimple Modules & Rings (Wedderburn-Artin Theory)

2.1 Simple Modules

$M \in \text{Mod-}R$ is *cyclic* if $\exists m \in M$: $M = mR$

$\Leftrightarrow$ $\exists$ epimorphism $\varphi : R_R \to M_R$ ["$\Rightarrow$": $r \mapsto mr$ "$\Leftarrow$" If $\varphi : R_R \to M_R$ is an epi, then $M = \varphi(R) = \underbrace{\varphi(1)}_{=: m} R$]

$\Leftrightarrow$ $M \cong R/I$ for a right ideal $I \leq R_R$ ["$\Rightarrow$": $M \cong R/\ker\varphi$

"$\Leftarrow$": $R \to R/I$, $r \mapsto r + I$ is epi]

Def: $M \in \text{Mod-}R$ is *simple* (= irreducible) if $M \neq 0$ and M has no proper nonzero submodule

Exm: •) The simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$, p prime

•) $R = k$ field: k_k is (up to isomorphism) the unique simple module

•) k field, $R = M_d(k)$, $V = k^{1 \times d}$ (row vectors) with R acting on the right. V is simple: for any $0 \neq v, w \in V$ $\exists A \in M_d(k)$ s.t. $vA = w$. So $vR = V$ $\forall v \in V \setminus \{0\}$.

Prop 2.1 For $M \in \text{Mod-}R$, TFAE:

(a) M is simple

(b) $M \neq 0$ and $\forall m \in M \setminus \{0\}: M = mR$. In particular: M is cyclic

(c) $M \cong R/I$ for a maximal right ideal

Proof: (a) $\Rightarrow$ (b) $M \neq 0$ ✓ Let $0 \neq m \in M \Rightarrow 0 \neq mR \leq M_R$
$\Rightarrow M = mR$

(b) $\Rightarrow$ (c) Let $0 \neq m \in M$. Let $\varphi: R_R \twoheadrightarrow M_R$, $r \mapsto mr$.

annihilator of m

$I := \ker(I) =: \mathrm{ann}(m)$

$\Rightarrow M \cong R/I$ with I a right ideal. If I is not maximal, there exists $x \in R$ s.t. $I \subsetneq I + xR \subsetneq R$. Then $\varphi(x) \neq 0$, and the cyclic module $\varphi(x)R$ is a proper submodule of M ↯.

(c) $\Rightarrow$ (a) Since $R/I \cong M$, the submodules of M are in bijection with right ideals J for which $I \subseteq J \subseteq R$.

Since $J = I$ or $J = R$, M is simple. $\square$

<u>Lemma 2.2</u> (Schur's lemma) Let $M, N \in \text{Mod-}R$ be simple.

If $f \in \mathrm{Hom}(M, N)$ then $f = 0$ or f is an isomorphism.

In particular: $\mathrm{End}(M_R)$ is a division ring.

<u>Proof</u>: $\ker f \leq M$, $\mathrm{im}\, f \leq N$, so $\ker f \in \{0, M\}$, $\mathrm{im}\, f \in \{0, N\}$

If $f \neq 0$, then $\ker f \neq M$, $\mathrm{im}\, f \neq \underline{0}$

$\Rightarrow \ker f = \underline{0}$, $\mathrm{im}\, f = N \Rightarrow f$ is an isomorphism. $\square$

<u>Remark</u>: If M, N are simple, either $M \cong N$ or $\mathrm{Hom}(M, N) = \underline{0}$.

<u>Recall</u>: A field k is alg. closed if every nonconstant $f \in k[x]$ has a root in k

$\Leftrightarrow$ Every $f \in k[x] \setminus k$ factors into linear factors

$\Leftrightarrow$ If L/k is a finite field extension, then $L = k$

Lemma 2.3 A field k is alg. closed $\Leftrightarrow$ If $D \supseteq k$ is a fin. dim division algebra $/k$ [i.e. div. ring & fin. dim k-algebra], then $D = k$.

Proof: "$\Leftarrow$" If L/k is a finite field ext., it is a fin. dim. div. algebra $/k$.

"$\Rightarrow$": Let $a \in D$. Since $k \subseteq Z(D)$, $k(a)/k$ is a field extension.

$\dim_k k(a) \leq \dim_k D < \infty \xRightarrow{k \text{ alg. closed}} k(a) = k \Rightarrow a \in k$. $\square$

Cor 2.4 Suppose k is an alg. closed field, R k-algebra, M a simple R-module s.t. $\dim_k M < \infty$. Then $\mathrm{End}(M_R) = k$ (canonically).

Proof: $k \hookrightarrow \mathrm{End}(M_R)$ via scalar mult.: $\lambda \in k \mapsto (m \mapsto m\lambda)$

$\mathrm{End}(M_R) \subseteq \mathrm{End}(M_k) \cong M_d(K)$ for $d = \dim_k M$, so $\mathrm{End}(M_R)$ is a fin-dim. k-algebra & a division ring (L 1.2).

$\xRightarrow{k \text{ alg. closed}} \mathrm{End}(M_R) = k$. $\square$

Example: If R is a fin. dim. $\mathbb{C}$-algebra (e.g. $R = \mathbb{C}[G]$ with G finite), $\mathrm{End}(M_R) \cong \mathbb{C}$ for all simple R-modules.

2.2. Composition Series

Def: Let $M \in \text{Mod-}R$. A *composition series* for M is a chain of submodules $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M$ s.t. M_i / M_{i-1} is simple for $1 \leq i \leq n$.

n is the *length* of the chain.

Exm: •) $\underset{\substack{\| \\ 3\cdot 2^2}}{(\mathbb{Z}/12\mathbb{Z})_{\mathbb{Z}}}$ has a c.s.: $0 = 12\mathbb{Z}/12\mathbb{Z} \overset{\mathbb{Z}/2\mathbb{Z}}{\subsetneq} 6\mathbb{Z}/12\mathbb{Z} \overset{\mathbb{Z}/3\mathbb{Z}}{\subsetneq} 2\mathbb{Z}/12\mathbb{Z} \overset{\mathbb{Z}/2\mathbb{Z}}{\subsetneq} \mathbb{Z}/12\mathbb{Z}$

•) $\mathbb{Z}_{\mathbb{Z}}$ does not have a composition series.

(between $n\mathbb{Z}$ & 0, we can always insert $mn\mathbb{Z}$, $m \geq 1$)

Lemma 2.5 (Modular Law) Let $M \in \text{Mod-}R$, $A, B, C \leq M_R$ s.t. $B \subseteq A$. Then
$$A \cap (B + C) = \underset{\substack{\| \\ (A\cap B)}}{B} + (A \cap C)$$

Proof: "$\supseteq$": $B \subseteq A \cap (B+C)$ and $(A \cap C) \subseteq A \cap (B+C)$ ✓

"$\subseteq$": Let $a = b + c$ with $a \in A$, $b \in B$, $c \in C$.

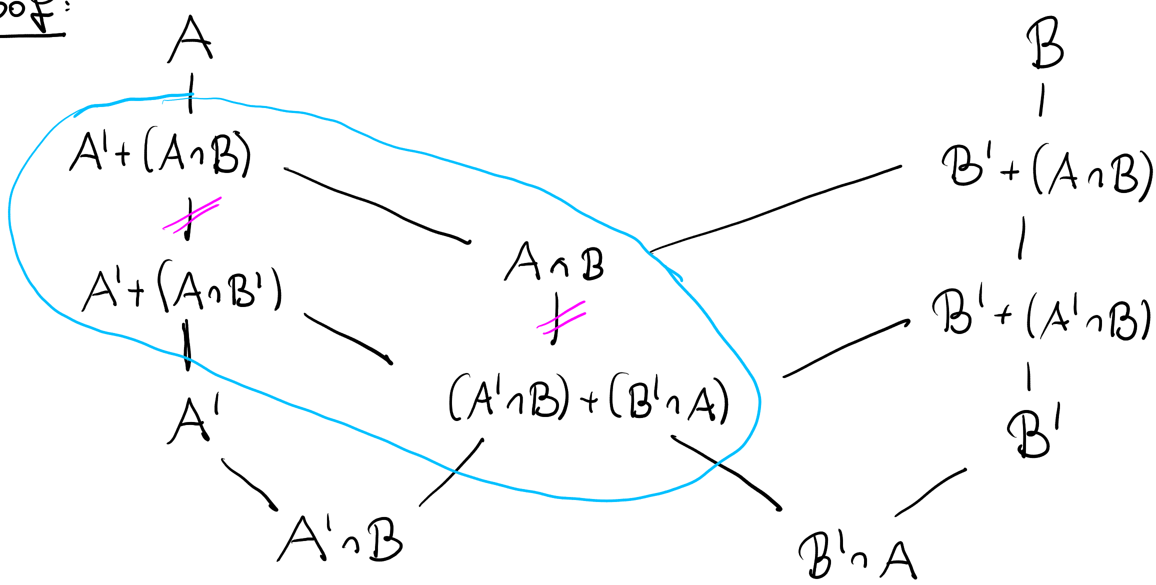
$\Longrightarrow c = a - b \in A \Longrightarrow a = \underset{\substack{\cap \\ B}}{b} + \underset{\substack{\cap \\ A\cap C}}{c} \in B + (A \cap C)$

Recall: If $A, B \leq M_R$ ($M \in \text{Mod-}R$), then
$$A+B/A \cong B/A\cap B$$

$$\begin{tikzcd}[column sep=small] & A+B \arrow[dl, dash, "//"'] \arrow[dr, dash] & \\ A \arrow[dr, dash] & & B \arrow[dl, dash, "//"] \\ & A\cap B & \end{tikzcd}$$

Lemma 2.6 (Zassenhaus) Let $A' \leq A$, $B' \leq B$ be submodules of some $M \in \text{Mod-}R$. Then
$$(A \cap B) + A' / (A \cap B') + A' \cong (A \cap B) + B' / (A' \cap B) + B'$$

Proof:



Note:

- $A' + \cancel{(A\cap B')} + (A\cap B) = A' + (A\cap B)$
- $(A\cap B) \cap (A' + (A\cap B')) \underset{A\cap B' \subseteq A\cap B}{\overset{L2.5}{=}} (\cancel{A}\cap B\cap A') + (A\cap B')$

$$= (B\cap A') + (A\cap B')$$

$$\Longrightarrow \quad \frac{A' + (A\cap B)}{A' + (A\cap B')} \cong \frac{A\cap B}{(B\cap A') + (A\cap B')} \underset{\text{Symmetry}}{\overset{\uparrow}{\cong}} \frac{B' + (A\cap B)}{B' + (A'\cap B)}$$

$\square$

Def: Two chains of submodules $0 = A_0 \leq A_1 \leq \dots \leq A_m = M$, $0 = B_0 \leq B_1 \leq \dots \leq B_n = M$ are equivalent (or isomorphic) if $m = n$ and there is a permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ s.t. $A_i / A_{i-1} \cong B_{\sigma(i)} / B_{\sigma(i)-1}$ $\forall i$.

Thm 2.7 (Schreier refinement theorem) Let $M \in \text{Mod-}R$. Any two chains

$$\underline{0} = A_0 \leq A_1 \leq \dots \leq A_m = M, \qquad \underline{0} = B_0 \leq B_1 \leq \dots \leq B_n = M$$

have equivalent refinements.

Proof: For $1 \leq i \leq m$, $1 \leq j \leq n$:

$$A_{i,j} := (A_i \cap B_j) + A_{i-1}$$

$$B_{j,i} := (A_i \cap B_j) + B_{j-1}$$

$\rightarrow \{A_{i,j} : j\}$ refines $A_{j-1} \leq A_j$, $\{B_{j,i} : i\}$ refines $B_{j-1} \leq B_j$:

$$A_i = A_{i,n} \geq A_{i,n-1} \geq \dots \geq A_{i,1} \geq A_{i,0} = A_{i-1}$$

$$B_j = B_{j,m} \geq B_{j,m-1} \geq \dots \geq B_{j,1} \geq B_{j,0} = B_{j-1}.$$

Now:

$$A_{i,j}/A_{i,j-1} \cong \frac{(A_i \cap B_j) + A_{i-1}}{(A_i \cap B_{j-1}) + A_{i-1}} \overset{L2.6}{\cong} \frac{(A_i \cap B_j) + B_{j-1}}{(A_{i-1} \cap B_j) + B_{j-1}}$$

$$\cong B_{j,i}/B_{j,i-1}$$

$\Rightarrow$ The refinements are equivalent. $\square$